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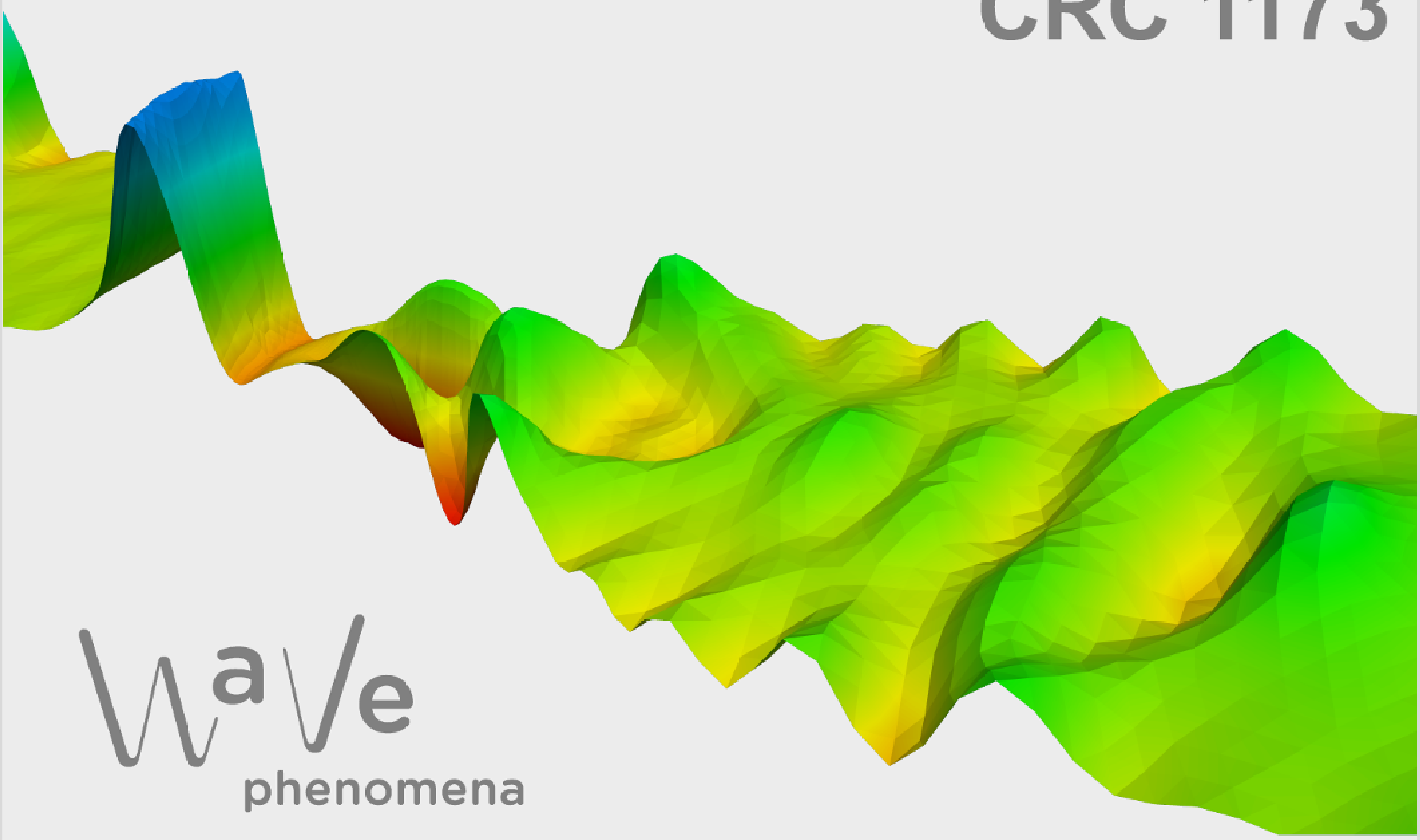

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VARIATIONAL METHODS FOR BREATHER SOLUTIONS OF NONLINEAR WAVE EQUATIONS

RAINER MANDEL AND DOMINIC SCHEIDER

Abstract. We construct infinitely many real-valued, time-periodic breather solutions of the nonlinear wave equation

$$\partial_t^2 U - \Delta U = Q(x)|U|^{p-2}U \quad \text{on } \mathbb{T} \times \mathbb{R}^N$$

with suitable $N \geq 2$, $p > 2$ and localized nonnegative Q. These solutions are obtained from critical points of a dual functional and they are weakly localized in space. Our abstract framework allows to find similar existence results for the Klein-Gordon equation or biharmonic wave equations.

1. Introduction

Breathers are real-valued, time-periodic and spatially localized solutions of nonlinear equations describing the propagation of waves on $\mathbb{R}^N \times \mathbb{R}$ where $N \in \mathbb{N}$. The existence of breather solutions appears to be rare phenomenon and up to now, most work in this area is related to the discussion of explicit examples such as the famous sine-Gordon breather for the $(1+1)$-sine-Gordon equation [1]. A number of (in-)stability results for such explicit breathers [2–5] is available. Nonexistence results can be found in [9, 22, 27]. The construction of non-explicit breather solutions is a very difficult task. In papers by Hirsch, Reichel [18, Theorem 1.3] and Blank, Chirilus-Bruckner, Lescarret, Schneider [6] this was achieved for nonlinear wave equations of the form

$$s(x)\partial_{tt}u - u_{xx} + q(x)u = f(x,u) \qquad (x, t \in \mathbb{R}) \tag{1}$$

following two completely different approaches. The methods from [6] come from spatial dynamics and rely on center manifold reductions. For one very specific choice of periodic step functions s, q (multiples of each other) and the nonlinearity $f(x,u) = u^3$, the authors prove the existence of L^∞-small periodic breather solutions that are exponentially localized in space. The particular choice for s and q is motivated by the underlying spectral theory of periodic Hill operators, also called Floquet theory. Using variational methods instead, Hirsch and Reichel [18] proved the existence of (spatially) square integrable breather solutions under appropriate assumptions on the nonlinearity. The latter include power-type nonlinearities $f(x,u) = |u|^{p-1}u$ with $1 < p < p^*$ for some p^* depending on the choice of s and q. Again, the potentials s, q are of very special form in order to ensure suitable spectral properties. More precisely, it is required that for all $k \in \mathbb{Z}$ the spectrum of the linear operator associated with the k-th mode does not contain 0 in a uniform sense. This makes it possible to have

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a strong localization in space. All of these results concern the case of one spatial dimension $N = 1$. The Bethe-Sommerfeld Conjecture about the number of gaps of periodic Schrödinger operators (see [23, Section 6.1.3]) suggests that the above approach can hardly be generalized to higher space dimensions that we discuss next.

In the case $N \geq 2$ we are aware of very few results. The first deals with a semilinear curl-curl wave equation in $\mathbb{R}^3 \times \mathbb{R}$ where $-u_{xx}$ is replaced by $\nabla \times \nabla \times u$ in (1) and u is a three-dimensional vector field on $\mathbb{R}^3$. Using that this part in the equation actually vanishes for gradient fields, Plum and Reichel [28] succeed in proving the existence of exponentially localized breather solutions via ODE methods for suitable radially symmetric coefficient functions s, q and power-type nonlinearities f. As far as we know, this is the only result dealing with strongly localized breathers in higher dimensions, i.e., $U(t, \cdot) \in L^2(\mathbb{R}^N)$ for almost all $t \in \mathbb{R}$. Recently, the second author suggested a new construction of (even in time) breathers [29] for the cubic Klein-Gordon equation that we will refer to as weakly localized in space. Those satisfy $U(\cdot, t) \in L^q(\mathbb{R}^N)$ for almost all $t \in \mathbb{R}$ for some $q > 2$ and we believe that in general $U(t, \cdot) \notin L^2(\mathbb{R}^N)$ holds due to rather small decay rates at infinity, presumably $U(t, x) \sim |x|^{(1-N)/2}$ as $|x| \to \infty$. That approach relies on the L^p-theory for Helmholtz equations on $\mathbb{R}^N$ and bifurcation techniques allow to prove the existence of infinitely many branches consisting of polychromatic radially symmetric breather solutions that emanate from a nontrivial stationary solution of the problem. The solutions are of the form

$$U(t, x) = \sum_{k \in \mathcal{I}_s} \mathrm{e}^{\mathrm{i}kt} u_k(x), \qquad \mathcal{I}_s \subseteq \mathbb{Z}$$

with radially symmetric Fourier modes $u_k = \overline{u_{-k}}$ infinitely many of which are non-zero. Imposing radial symmetry is of course a significant restriction. In this paper we propose a variational approach for the construction of weakly localized breather solutions that does not rely on symmetry assumptions for the coefficients and generalizes to much more general classes of equations. Before going into the details of our method, we would like to point that it would be most desirable to find strongly localized breather solutions for nonlinear wave-type equations in the case of spatial dimensions $N \geq 2$.

Here we prove the existence of weakly localized breathers for wave equations of the form

$$\partial_t^2 U - \mathcal{L} U = Q(x)|U|^{p-2}U \quad \text{on } \mathbb{R}^N \times \mathbb{R}. \tag{2}$$

The data Q, p will be chosen suitably in dependence of the mapping properties of certain right inverses of $\mathcal{L}$. As a model case one may have in mind $\mathcal{L} = \Delta$. In contrast to [29] we can deal with nonradial Q and obtain the existence of an unbounded sequence of breathers. In order to avoid "bad" modes, we look for breather solutions that may enjoy additional symmetry properties with respect to time. To include such symmetries in our analysis we introduce a parameter $s \in \{1, \dots, 5\}$ that stands for

- $(s = 1)$ no additional symmetry,
- $(s = 2)$ $U(t, x) = U(-t, x)$, i.e., U is even in time,
- $(s = 3)$ $U(t, x) = -U(-t, x)$, i.e., U is odd in time,
- $(s = 4)$ $U(t + \pi, x) = U(t, x)$, i.e., U is π-periodic,

$(s=5)\quad U(t+\pi,x)=-U(t,x)$, i.e., U is π-antiperiodic.

For those symmetries the relevant modes $k\in\mathcal{I}_s\subset\mathbb{Z}$ in the corresponding Fourier expansions come from the sets

$$\mathcal{I}_1:=\mathbb{Z},\qquad \mathcal{I}_2:=\mathbb{Z},\qquad \mathcal{I}_3:=\mathbb{Z}\setminus\{0\},\qquad \mathcal{I}_4:=2\mathbb{Z},\qquad \mathcal{I}_5:=2\mathbb{Z}+1. \tag{3}$$

As to $s=2$ resp. $s=3$ observe that the symmetry assumptions moreover imply that the modes u_k resp. iu_k are real-valued. Regarding $s=4$ let us mention that general periods $T>0$ can be discussed, as we will explain in Section 2.5. We will assume that the following conditions are satisfied for the modes $k\in\mathcal{I}_s$:

(A1) There are (real-valued) bounded symmetric operators $\mathcal{R}_k : L^{q'}(\mathbb{R}^N)\to L^q(\mathbb{R}^N)$ for some $q\in[p,\infty]$ that satisfy $\|\mathcal{R}_k\|\le C(k^2+1)^{-\frac{\alpha}{2}}$ for some $\alpha>1-\frac{2}{p}$ as well as
$$\int_{\mathbb{R}^N}\mathcal{R}_k f\cdot(-\mathcal{L}-k^2)\phi\,\mathrm{d}x=\int_{\mathbb{R}^N}f\phi\,\mathrm{d}x\qquad\text{for all } f\in L^{q'}(\mathbb{R}^N),\ \phi\in C_c^\infty(\mathbb{R}^N).$$

(A2) $Q\in L^{q/(q-p)}(\mathbb{R}^N), Q\ge 0, Q\not\equiv 0$ and the linear operators $v\mapsto\mathcal{R}_k^Q[v]:=Q^{1/p}\mathcal{R}_k[Q^{1/p}v]$ are compact from $L^{p'}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$.

(A3) There are $\omega_k\in L^{p'}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\omega_k\mathcal{R}_k^Q[\omega_k]\,\mathrm{d}x>0$.

We briefly comment on these assumptions. The operators $\mathcal{R}_k$ can be interpreted as distributional inverses of $-\mathcal{L}-k^2$ which may even exist when classical inverses are not available. This is for instance the case in our model example $\mathcal{L}=\Delta$ as we will show below. The symmetry assumption means

$$\int_{\mathbb{R}^N}\mathcal{R}_k f\cdot g\,\mathrm{d}x=\int_{\mathbb{R}^N}f\cdot\mathcal{R}_k g\,\mathrm{d}x\quad\text{for all } f,g\in L^{q'}(\mathbb{R}^N).$$

The growth bound on the norms ensures the convergence of Fourier series in topologies that are suitable for our analysis. Assumption (A2) is needed for our dual variational approach, notably for the verification of the Palais-Smale condition of the functional J to be introduced below. Notice that $Q\in L^{q/(q-p)}(\mathbb{R}^N), Q\ge 0$ and (A1) already imply the boundedness of $\mathcal{R}_k^Q$ as a map from $L^{p'}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$, see (12). For $\mathcal{L}=\Delta$ such operators are called Birman-Schwinger operators in the literature. Finally, (A3) is a technical assumption that holds in many applications. It is for instance satisfied if Q is positive and for all $k\in\mathcal{I}_s$ there are test functions $\phi_k\in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}\phi_k(-\mathcal{L}-k^2)\phi_k\,\mathrm{d}x>0$. This follows from the choice $\omega_k:=Q^{-1/p}1_{Q>\delta}(-\mathcal{L}-k^2)\phi_k\in L^{p'}(\mathbb{R}^N)$ for sufficiently small $\delta>0$. Accordingly, this assumption always holds for $Q>0$ and elliptic differential operators $\mathcal{L}$. In Section 2 we will see a number of settings where all these assumptions hold.

Assuming (A1)-(A3) for all modes $k\in\mathcal{I}_s$ we are going to prove the existence of breather solutions U of (2) in the Banach space $L^q(\mathbb{R}^N,L^p_s(\mathbb{T}))$ consisting of all elements of $L^q(\mathbb{R}^N,L^p(\mathbb{T}))$ having the symmetry indexed by s. Here, $\mathbb{T}\simeq[0,2\pi]$ stands for the torus. The norm on these spaces is given by

$$\|W\|_{L^q(\mathbb{R}^N,L^p(\mathbb{T}))}:=\Big\|\|W(\cdot,x)\|_{L^p(\mathbb{T})}\Big\|_{L^q(\mathbb{R}^N)}:=\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{T}}|W(t,x)|^p\,\mathrm{d}t\right)^{q/p}\mathrm{d}x\right)^{1/q}.$$

More precisely, we will speak of 2π-periodic distributional breather solutions since we require these functions to solve equation (2) in the following sense:

$$\int_{\mathbb{T}\times\mathbb{R}^N} U\,(\partial_t^2-\mathcal{L})\Phi\,\mathrm{d}(t,x)=\int_{\mathbb{T}\times\mathbb{R}^N} Q(x)|U|^{p-2}U\,\Phi\,\mathrm{d}(t,x)\quad \forall\Phi\in C_c^\infty(\mathbb{R}^N,C^\infty(\mathbb{T})). \tag{4}$$

Here, $\Phi\in C_c^\infty(\mathbb{R}^N,C^\infty(\mathbb{T}))$ means that there is a compact subset $K\subseteq\mathbb{R}^N$ such that $\Phi:\mathbb{R}^N\times\mathbb{T}\to\mathbb{R}$ is smooth, 2π-periodic in time and the support of $\Phi(t,\cdot)$ is contained in K for all $t\in\mathbb{T}$. Our main result is the following.

Theorem 1. *Assume $N\in\mathbb{N}$, $2<p<\infty$ and (A1)-(A3) for all $k\in\mathcal{I}_s$ where $s\in\{1,\dots,5\}$. Then the nonlinear wave equation* (2) *admits an unbounded sequence of 2π-periodic distributional breather solutions $U_j\in L^q(\mathbb{R}^N,L^p_s(\mathbb{T}))$, $j\in\mathbb{N}_0$, in the sense of* (4).

Let us add that our breather solutions are either constant or polychromatic. The latter means that at least two Fourier modes u_k,u_l with $k\neq l,k,l\in\mathbb{N}_0$ of the solution are nonzero. Indeed, plugging the ansatz $U(x,t)=\cos(kt)u_k(x)$ or $U(x,t)=\sin(kt)u_k(x)$ for some $k\in\mathcal{I}_s$ and nontrivial functions u_k into (2) one infers $k=0$, so U is necessarily constant in time. In our applications below this will be avoided by choosing $s\in\{3,5\}$ because of $0\notin\mathcal{I}_s$. In the case $p\in\{3,4,\dots\}$ we can conclude as in [29, Theorem 1 (iii)] that nonconstant in time breathers have infinitely many nontrivial modes, which we believe to be the typical situation.

We first outline how this paper is organized. In Section 2 we show how Theorem 1 applies in concrete situations. In particular, we prove the existence of infinitely many breathers of nonlinear wave equations and Klein-Gordon equations on $\mathbb{R}^N$. Moreover, we indicate further possible generalizations of our approach. In Section 3 we motivate our variational approach and present the proof of Theorem 1 relying on the technical results contained in the Propositions 2–4. The proofs of the latter are presented in Section 4.

2. Applications and Examples

2.1. Breather Solutions for the Wave Equation on $\mathbb{R}^N$.

We show that Theorem 1 applies to classical nonlinear wave equations on $\mathbb{R}^N$ with power-type nonlinearities

$$\partial_t^2U-\Delta U=Q(x)|U|^{p-2}U\qquad \text{on } \mathbb{T}\times\mathbb{R}^N. \tag{5}$$

To verify (A1)–(A3) we need distributional inverses for operators of the form $-\Delta-k^2$ for $k\in\mathcal{I}_s$ and suitable $s\in\{1,\dots,5\}$. From [21, Theorem 2.3], [17, Theorem 6] ($N\geq 3$) and [12, Theorem 2.1] ($N=2$) we infer that that the operators

$$\mathcal{R}_kf:=\lim_{\varepsilon\to 0^+}\operatorname{Re}\left[\mathcal{F}^{-1}\left(\frac{\mathcal{F}f}{|\cdot|^2-k^2-i\varepsilon}\right)\right]$$

are suitable for that purpose. Here, $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^N$. For the asymptotics with respect to k we refer to [21, Theorem 2.3] ($N\geq 3$) and inequality (8) in [14] ($N=2$).

Lemma 1. *Let $N \in \mathbb{N}, N \geq 3$ and assume $\frac{2(N+1)}{N-1} \leq r \leq \frac{2N}{N-2}$. For every $k \in \mathbb{Z} \setminus \{0\}$ the operator $\mathcal{R}_k : L^{r'}(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ is a real-valued, bounded and symmetric distributional inverse of $-\Delta - k^2$ satisfying*

$$\|\mathcal{R}_k f\|_r \leq C|k|^{-2+\frac{N}{r'}-\frac{N}{r}} \|f\|_{r'}$$

for some $C > 0$. In the case $N = 2$ the same holds for $6 \leq r < \infty$.

We stress that this result does not provide an inverse for $-\Delta$, which is why we have to consider symmetries that exclude the zero mode. This and (3) motivate the choice $s \in \{3, 5\}$, so that we obtain the existence of infinitely many odd-in-time 2π-periodic breathers and infinitely many π-antiperiodic breathers.

Corollary 1 (The Wave Equation)**.** *Assume $N \in \mathbb{N}, N \geq 2$ and $Q \in L^{\frac{q}{q-p}}(\mathbb{R}^N), Q \geq 0, Q \not\equiv 0$ where p, q satisfy*

$$2 < p < \frac{2(N+1)}{N-1}, \qquad \frac{2(N+1)}{N-1} < q < \frac{2Np}{(N-1)p-2}.$$

Then, for $s \in \{3, 5\}$, the wave equation (5) admits an unbounded sequence of 2π-periodic distributional breather solutions $U_j \in L^q(\mathbb{R}^N; L^p_s(\mathbb{T}))$, $j \in \mathbb{N}_0$.

Proof of Corollary 1. We verify the assumptions (A1) - (A3) for $\mathcal{L} = \Delta$. As indicated above the choice $s \in \{3, 5\}$ implies $\mathcal{I}_s \subset \mathbb{Z} \setminus \{0\}$ so that $k \in \mathcal{I}_s$ implies $|k| \geq 1$. In particular, the previous lemma applies and yields real-valued, bounded, symmetric linear operators $\mathcal{R}_k$ that are distributional inverses of $-\Delta - k^2$ and satisfy

$$\|\mathcal{R}_k f\|_q \leq C(k^2 + 1)^{-\alpha/2} \|f\|_{q'} \qquad (k \in \mathcal{I}_s)$$

for $\alpha = 2 - \frac{N}{q'} + \frac{N}{q}$. Here we have used that our assumptions imply $\frac{2(N+1)}{N-1} \leq q < \frac{2N}{N-2}$. From $q < \frac{2Np}{(N-1)p-2}$ we moreover infer $\alpha > 1 - \frac{2}{p}$. So assumption (A1) holds. The compactness of the Birman-Schwinger operator $\mathcal{R}_k^Q$ is proved as in [13, Lemma 4.1] ($N \geq 3$) resp. [12, Section 3] ($N = 2$) or [16, Lemma 3.1]. (We will provide more details in the proof of Corollary 3 below.) Taking $\omega_k := v_0(k\cdot)$ for the function v_0 from [13, Lemma 4.2 (ii)] we find that (A3) holds as well. Hence, Theorem 1 yields the existence of and unbounded sequence of distributional breathers in $L^q(\mathbb{R}^N; L^p_s(\mathbb{T}))$. □

2.2. **Breather Solutions for the Klein-Gordon Equation.** We study the Klein-Gordon equation

$$\partial_t^2 U - \Delta U + m^2 U = Q(x)|U|^{p-2}U \qquad \text{on } \mathbb{T} \times \mathbb{R}^N. \tag{6}$$

Much like for the wave equation, we deduce from Theorem 1 the following

Corollary 2 (The Klein-Gordon Equation)**.** *Assume $N \in \mathbb{N}, N \geq 2$, $m > 0$ and $Q \in L^{\frac{q}{q-p}}(\mathbb{R}^N), Q \geq 0, Q \not\equiv 0$ where p, q satisfy*

$$2 < p < \frac{2(N+1)}{N-1}, \qquad \frac{2(N+1)}{N-1} < q < \frac{2Np}{(N-1)p-2}.$$

Then the Klein-Gordon equation (6) *admits an unbounded sequence of* 2π*-periodic breather solutions* $U_j \in L^q(\mathbb{R}^N; L^p_s(\mathbb{T}))$, $j \in \mathbb{N}_0$. *Here,* s *can be chosen as follows:*

(i) *if* $m \notin \mathbb{N}$, *then* $s \in \{1, \dots, 5\}$,

(ii) *if* $m \in 2\mathbb{N} - 1$, *then* $s = 4$ (π*-periodic breathers),*

(iii) *if* $m \in 2\mathbb{N}$, *then* $s = 5$ (π*-antiperiodic breathers).*

Since the proof is very much the same as for the wave equation, we omit it. Let us remark that our choice of s again ensures that we avoid the modes $k \in \mathcal{I}_s$ with $m^2 - k^2 = 0$. In the study of the operators $-\Delta + m^2 - k^2$, there may now occur a finite number of operators ($k \in \mathcal{I}_s$ with $k^2 < m^2$) with classical L^2-inverses given by a convolution with positive exponentially decaying kernels. These well-understood Bessel potential operators also satisfy (A1)–(A3). Notice that their mapping properties are even better than the ones mentioned in Lemma 1 because all $r \in [2, \frac{2N}{N-2}]$ are allowed if $N \geq 3$.

2.3. **Breather Solutions for Fractional and Biharmonic Wave Equations.** We consider the problem

$$\partial_t^2 U + (-\Delta)^\gamma U = Q(x)|U|^{p-2}U \qquad \text{on } \mathbb{T} \times \mathbb{R}^N \tag{7}$$

for general $\gamma > \frac{N}{N+1}$. As in the case of the classical wave equation one finds distributional inverses of $(-\Delta)^\gamma - k^2$ with the aid of the Limiting Absorption Principle that allows to make sense of the limits

$$\mathcal{R}_k^\gamma f := \lim_{\varepsilon \to 0^+} \mathrm{Re}\left[\mathcal{F}^{-1}\left(\frac{\mathcal{F}f}{|\cdot|^{2\gamma} - k^2 - i\varepsilon}\right)\right].$$

This follows from a result by Huang [20, Corollary 3.2].

Lemma 2 (Huang)**.** *Let* $N \in \mathbb{N}, N \geq 3$ *and assume* $\frac{2(N+1)}{N-1} \leq r < \frac{2N}{(N-2\gamma)_+}, \gamma \geq \frac{N}{N+1}$. *For every* $k \in \mathbb{Z} \setminus \{0\}$ *the operator* $\mathcal{R}_k^\gamma : L^{r'}(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ *is a real-valued, bounded and symmetric distributional inverse of* $(-\Delta)^\gamma - k^2$ *satisfying*

$$\|\mathcal{R}_k^\gamma f\|_r \leq |k|^{-2+\frac{N}{\gamma r'}-\frac{N}{\gamma r}}\|f\|_{r'}.$$

The previous lemma provides the existence of distributional inverses for all $\gamma \geq \frac{N}{N+1}$. As in the case of the Laplacian we expect a similar result to hold in the two-dimensional case $N = 2$. In the following result we apply Lemma 2 in order to prove the existence of breathers to fractional nonlinear wave equations just as in the case $\gamma = 1$ discussed in Corollary 1. We stress that this includes the case $\gamma = 2$ of a biharmonic wave equation.

Corollary 3 (Fractional Wave Equations)**.** *Assume* $N \in \mathbb{N}, N \geq 3, \gamma > \frac{N}{N+1}$ *and* $Q \in L^{\frac{q}{q-p}}(\mathbb{R}^N), Q > 0$ *where* p, q *satisfy*

$$2 < p < \frac{2\gamma(N+1)}{((2-\gamma)N - \gamma)_+}, \qquad \frac{2(N+1)}{N-1} < q < \frac{2Np}{((N-\gamma)p - 2\gamma)_+}.$$

Then, for $s \in \{3, 5\}$, *the fractional wave equation* (7) *admits an unbounded sequence of* 2π*-periodic distributional breather solutions* $U_j \in L^q(\mathbb{R}^N, L^p_s(\mathbb{T}))$, $j \in \mathbb{N}_0$.

Proof of Corollary 3. As in the proof of Corollary 1 the previous lemma yields assumption (A1) for $\alpha = 2 - \frac{N}{\gamma q'} + \frac{N}{\gamma q}$. For the verification of (A3) we follow [25, Lemma 3.1]. We define $\omega_k^\delta := Q^{-1/p}\tilde{\omega}_k 1_{Q\geq\delta} \in L^{p'}(\mathbb{R}^N)$ where $\delta > 0$ is sufficiently small and the support of $\mathcal{F}(\tilde{\omega}_k)$ is contained in $\{\xi \in \mathbb{R}^N : |\xi|^{2\gamma} > k^2\}$. Then the above definition for $\mathcal{R}_k^\gamma$ implies

$$\begin{aligned}\lim_{\delta\to 0^+}\int_{\mathbb{R}^N} \omega_k^\delta Q^{1/p}\mathcal{R}_k^\gamma[Q^{1/p}\omega_k^\delta]\,\mathrm{d}x &= \int_{\mathbb{R}^N} \tilde{\omega}_k\mathcal{R}_k^\gamma[\tilde{\omega}_k]\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\mathrm{Re}\left(\int_{\mathbb{R}^N}\frac{|\mathcal{F}(\tilde{\omega}_k)|^2}{|\xi|^{2\gamma}-k^2-i\varepsilon}\,\mathrm{d}\xi\right) \\ &= \int_{\mathbb{R}^N}\frac{|\mathcal{F}(\tilde{\omega}_k)|^2}{|\xi|^{2\gamma}-k^2}\,\mathrm{d}\xi > 0.\end{aligned}$$

So choosing $\delta > 0$ sufficiently yields (A3).

The verification of (A2) is standard for classical Schrödinger operators of second order. In order to see that in the fractional case nothing really changes, we repeat the main arguments here. In view of

$$\|Q^{1/p}\mathcal{R}_k^\gamma[Q^{1/p}v]\|_p \leq \|Q\|_{\frac{q}{q-p}}^{1/p}\|\mathcal{R}_k^\gamma[Q^{1/p}v]\|_q$$

it suffices to prove that $v \mapsto \mathcal{R}_k^\gamma[\Gamma v]$ is compact from $L^{p'}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ where $\Gamma := Q^{1/p}$. We may without loss of generality assume that Γ is bounded with compact support. Indeed, choosing $\Gamma_n \to \Gamma$ in $L^{\frac{pq}{q-p}}(\mathbb{R}^N)$ with Γ_n bounded and compact support, we find

$$\|\mathcal{R}_k^\gamma[\Gamma v] - \mathcal{R}_k^\gamma[\Gamma_n v]\|_q \leq \|\mathcal{R}_k^\gamma[(\Gamma-\Gamma_n)v]\|_q \leq \|\mathcal{R}_k^\gamma\|_{q'\to q}\underbrace{\|\Gamma-\Gamma_n\|_{\frac{pq}{q-p}}}_{\to 0 \text{ as } n\to\infty}\|v\|_{p'}.$$

Having proved that $v \mapsto \mathcal{R}_k^\gamma[\Gamma_n v]$ is compact for each $n \in \mathbb{N}$ we can thus conclude that $v \mapsto \mathcal{R}_k^\gamma[\Gamma v]$ is compact as the limit of compact operators with respect to the uniform operator topology. So it remains to prove the compactness of $v \mapsto \mathcal{R}_k^\gamma[\Gamma v]$ from $L^{p'}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ assuming that Γ is bounded with compact support.

Let $B \subset \mathbb{R}^N$ be any bounded ball. The compactness of $v \mapsto \chi_B\mathcal{R}_k^\gamma[\Gamma v]$ follows from the fractional Rellich-Kondrachov Theorem, see [10, Corollary 7.2]. By the same argument as above, it remains to show $\|\chi_{\mathbb{R}^N\setminus B}\mathcal{R}_k^\gamma[\Gamma\cdot]\|_{p'\to q} \to 0$ as $B \nearrow \mathbb{R}^n$. To this end we use

$$\mathcal{R}_k^\gamma f = G_k^\gamma * f, \qquad \text{where } G_k^\gamma(z) := \lim_{\varepsilon\to 0^+}\mathrm{Re}\left[\mathcal{F}^{-1}\left(\frac{1}{|\cdot|^{2\gamma}-k^2-i\varepsilon}\right)(z)\right].$$

The formulas (3.8),(3.8') in [20, Corollary 3.2] show that the kernel function satisfies $|G_k^\gamma(z)| \leq C_k|z|^{\frac{1-N}{2}}$ if $|z| \geq 1$ for some $C_k > 0$. Hence, for $M := \mathrm{supp}(\Gamma)$ adn $x \in \mathbb{R}^N$ such that $\mathrm{dist}(x, M) \geq 1$ we have

$$|\mathcal{R}_k^\gamma[\Gamma v](x)| \leq C_k\int_M |x-y|^{\frac{1-N}{2}}|\Gamma(y)||v(y)|\,\mathrm{d}y \leq \tilde{C}_k|x|^{\frac{1-N}{2}}\|\Gamma\|_p\|v\|_{p'}.$$

This yields for large enough balls B

$$\|\chi_{\mathbb{R}^N\setminus B}\mathcal{R}_k^\gamma[\Gamma v]\|_q \leq C\|\Gamma\|_p\|v\|_{p'}\Big(\int_{\mathbb{R}^N\setminus B}|x|^{\frac{q(1-N)}{2}}\,\mathrm{d}x\Big)^{\frac{1}{q}}$$

and the conclusion follows due to $q > \frac{2(N+1)}{N-1} > \frac{2N}{N-1}$. $\square$

2.4. **Breather Solutions for the perturbed Wave Equation.** We consider

$$\partial_t^2 U - \Delta U + V(x)U = Q(x)|U|^{p-2}U \qquad \text{on } \mathbb{T}\times\mathbb{R}^N \tag{8}$$

where now V is a short-range potential. In a forthcoming paper by L. Cossetti and the first author the following generalization of Lemma 1 is proved.

Lemma 3 (Cossetti, Mandel). *Let $N\in\mathbb{N}, N\geq 3$ and assume $V\in L^{\frac{N}{2}}(\mathbb{R}^N)+L^{\frac{N+1}{2}}(\mathbb{R}^N)$ and $\frac{2(N+1)}{N-1}\leq r\leq \frac{2N}{N-2}$. For every $k\in\mathbb{Z}\setminus\{0\}$ there is a real-valued, bounded and symmetric distributional inverse $\mathcal{R}_k : L^{r'}(\mathbb{R}^N)\to L^r(\mathbb{R}^N)$ of $-\Delta+V(x)-k^2$ satisfying*

$$\|\mathcal{R}_k f\|_r \leq |k|^{-2+\frac{N}{r'}-\frac{N}{r}}\|f\|_{r'}.$$

Corollary 4. *Assume $N\in\mathbb{N}, N\geq 2$, V as in Lemma 3 and $Q\in L^{\frac{q}{q-p}}(\mathbb{R}^N), Q>0$ where p,q satisfy*

$$2<p<\frac{2(N+1)}{N-1}, \qquad \frac{2(N+1)}{N-1}<q<\frac{2Np}{(N-1)p-2}.$$

Then, for $s\in\{3,5\}$, the perturbed wave equation (8) admits an unbounded sequence of 2π-periodic distributional breather solutions $U_j\in L^q(\mathbb{R}^N;L^p_s(\mathbb{T}))$, $j\in\mathbb{N}_0$.

Proof. (A1) and (A2) can be found in [8]. To see that (A3) holds it remains to choose a test function $\phi_k\in C_c^\infty(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N}\phi_k(-\mathcal{L}-k^2)\phi_k \mathrm{d}x$. To this end we write $V=V_1+V_2$ with $V_1\in L^{N/2}(\mathbb{R}^N), V_2\in L^{(N+1)/2}(\mathbb{R}^N)$. Replacing V_1,V_2 by $V_1 1_{|V|>R}$ respectively $V_1 1_{|V|\leq R}+V_2$ if necessary we can assume that $\|V_1\|_{N/2}$ is small enough. So we obtain for every $\delta>0$ some $C_\delta>0$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\phi_k(-\mathcal{L}-k^2)\phi_k\,\mathrm{d}x\\
&=\int_{\mathbb{R}^N}|\nabla\phi_k|^2+(V(x)-k^2)|\phi_k|^2\,\mathrm{d}x\\
&\geq \|\nabla\phi_k\|_2^2-\left(\|V_1\|_{\frac{N}{2}}\|\phi_k\|^2_{\frac{2N}{N-2}}+\|V_2\|_{\frac{N+1}{2}}\|\phi_k\|^2_{\frac{2(N+1)}{N-1}}\right)-k^2\|\phi_k\|_2^2\\
&\geq \|\nabla\phi_k\|_2^2-\delta\|\phi_k\|^2_{\frac{2N}{N-2}}-(C_\delta+k^2)\|\phi_k\|_2^2\\
&\geq (1-C_S^2\delta)\|\nabla\phi_k\|_2^2-(C_\delta+k^2)\|\phi_k\|_2^2.
\end{aligned}$$

Here, $C_S>0$ comes from the Sobolev Embedding Theorem. Choosing $\phi=\phi^*(t_k\cdot)$ for some fixed nontrivial $\phi^*\in C_c^\infty(\mathbb{R}^N)$ and large enough $t_k>0$ we get the result. $\square$

2.5. **Generalizations.** Before going on with the proof of our main result we indicate further generalizations of our method.

- **(General periods)** In our main result we presented the theory for 2π-periodic breathers. Clearly, by rescaling, there is an analogous theory for T-periodic breathers for any given $T>0$. Analytically this does not change much, but explicit criteria in concrete applications need to be adapted. For instance, in Corollary 2 dealing with the Klein-Gordon equation the conditions on the mass m in (i),(ii),(iii) need to be replaced by the corresponding condition on $\frac{Tm}{2\pi}$.

- **(Negative Q)** In (A2) we assume nonnegative Q since this is required by the classical dual variational approach that we implement in our paper. In the context of the Helmholtz-type problems it is possible to deal with $Q \leq 0$ as well. Indeed, following [25, Section 3] we can slightly modify our functional J from (9) below to cover this case as well. In this way one obtains the existence of infinitely many breathers in that case. In the case of an elliptic operator $\mathcal{L}$, say $\mathcal{L} = \Delta$, the corresponding solutions are not stationary and hence polychromatic for all $s \in \{1, \ldots, 5\}$. This is true because the maximum principle implies that solutions of $-\Delta U = Q|U|^{p-2}U$ are necessarily trivial in this case. Sign-changing Q are much more delicate and up to now, dual variational techniques have not proven to be applicable.
- **(Non-Euclidean Settings)** Resolvent estimates of type $L^p - L^q$ also hold in the hyperbolic space, see [7, Theorem 2.3] and [19, Theorem 1.2]. The decay rate of the operator norms of the corresponding right inverses with respect to k, however, is not known as far as we can see. Once a bound as in (A1) is proved, our method yields the existence of breathers in this case, too.
- **(General evolutions)** The wave operator $\partial_t^2 - \mathcal{L}$ can be replaced by $P(-i\partial_t) - \mathcal{L}$ where $P : \mathbb{R} \to \mathbb{R}$ is a polynomial. In that case, the main difficulty is the construction of a distributional inverse of $-\mathcal{L} + P(k)$. One also has to review the definition of $\mathcal{I}_s$ and Proposition 2 (i) where the compatiblity of $\mathcal{R}_k$ with these symmetries is proved. The results for odd respectively even polynomials P will be different here.
- **(Systems)** One may ask whether breathers also exist for coupled nonlinear wave equations. Following our approach, this leads to infinite systems of coupled nonlinear Helmholtz systems. We believe that some ideas from the paper [26] about 2×2-Nonlinear Helmholtz Systems can be used.
- **(General nonlinearities)** Our paper deals with power-type nonlinearities, but the dual variational technique is actually a bit more flexible. To apply this method to a general nonlinearity $f(x, u)$ (replacing $Q(x)|u|^{p-2}u$ in (2)), one has to require the invertibility of $z \mapsto f(x, z)$ for almost all $x \in \mathbb{R}^N$. Very often this is guaranteed by imposing a monotonicity assumption with respect to z. Moreover, this inverse needs to give rise to a dual functional having the Mountain Pass Geoemtry on an appropriate Banach space. In some cases, if the nonlinearity does not behave like a power everywhere, Orlicz spaces can be used, see for instance [11]. Being interested in an unbounded sequence of breather solutions, one further has to impose that the nonlinearity is odd with respect to the second entry. Ideas for treating general nonlinearities or power-type nonlinearities $f(x, u) = Q(x)|u|^{p-2}u$ with $Q \in L^\infty(\mathbb{R}^n)$ using a fixed point approach can be found in [24].

3. Proof of Theorem 1

To motivate our variational approach we introduce the formal Fourier series expansion

$$U(t,x)=\sum_{k\in\mathcal{I}_s}\mathrm{e}^{\mathrm{i}kt}u_k(x)\quad\text{with Fourier modes } u_k(x):=[U]_k(x):=\frac{1}{2\pi}\int_{\mathbb{T}}\mathrm{e}^{-\mathrm{i}kt}U(t,x)\,\mathrm{d}t.$$

Recall that $\mathcal{I}_s$ collects the frequencies that are needed for building up breather solutions U with the symmetry indexed by $s\in\{1,\dots,5\}$ as in the Introduction. Plugging this ansatz into (2) we are lead to the infinite system of equations

$$(-\mathcal{L}-k^2)u_k=[Q|U|^{p-2}U]_k=Q^{1/p}\,[Q^{1/p'}|U|^{p-2}U]_k\qquad(k\in\mathcal{I}_s).$$

We introduce the dual variable $V:=Q^{1/p'}|U|^{p-2}U$ with formal Fourier series expansion

$$V(t,x)=\sum_{k\in\mathcal{I}_s}\mathrm{e}^{\mathrm{i}kt}v_k(x)\quad\text{with } v_k(x)=[V]_k(x)=\frac{1}{2\pi}\int_{\mathbb{T}}\mathrm{e}^{-\mathrm{i}kt}V(t,x)\,\mathrm{d}t.$$

So we can rewrite the above system as follows

$$\begin{aligned}(-\mathcal{L}-k^2)u_k=Q^{1/p}[Q^{1/p'}|U|^{p-2}U]_k\quad&\rightsquigarrow\quad Q^{1/p}u_k=\mathcal{R}_k^Q\left[[Q^{1/p'}|U|^{p-2}U]_k\right]\\&\rightsquigarrow\quad Q^{1/p}U(t,\cdot)=\sum_{k\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}kt}\mathcal{R}_k^Q\left[[Q^{1/p'}|U|^{p-2}U]_k\right]\\&\rightsquigarrow\quad |V(t,\cdot)|^{p'-2}V(t,\cdot)=\sum_{k\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}kt}\mathcal{R}_k^Q\,[v_k]\\&\rightsquigarrow\quad |V|^{p'-2}V=\mathscr{R}[V]\end{aligned}$$

where the operator $\mathscr{R}$ is defined via

$$\mathscr{R}[V](t,x):=\sum_{k\in\mathcal{I}_s}\mathrm{e}^{\mathrm{i}kt}\mathcal{R}_k^Q\,[v_k]\,(x)\quad\text{with } v_k(x)=[V]_k(x)=\frac{1}{2\pi}\int_{\mathbb{T}}\mathrm{e}^{-\mathrm{i}kt}V(t,x)\,\mathrm{d}t.$$

If Q is positive, then we can similarly derive the formula $U=Q^{-1/p}\mathscr{R}[V]$. So in order to deduce integrability properties of U, we will therefore study the operator

$$(Q^{-1/p}\mathscr{R})[V]:=\sum_{k\in\mathcal{I}_s}\mathrm{e}^{\mathrm{i}kt}\mathcal{R}_k\big[Q^{1/p}v_k\big],$$

which makes sense even under our weaker assumptions on Q, namely $Q\geq 0, Q\not\equiv 0$. Since $\mathscr{R}$ will turn out to be symmetric, the above equation for V has a variational structure. It is the Euler-Lagrange equation of the functional

$$J(V):=\frac{1}{p'}\int_{\mathbb{T}\times\mathbb{R}^N}|V|^{p'}\,\mathrm{d}(t,x)-\frac{1}{2}\int_{\mathbb{T}\times\mathbb{R}^N}V\mathscr{R}[V]\,\mathrm{d}(t,x)\tag{9}$$

so that we are lead to prove the existence of critical points. This motivates the following discussion of the functional J and finishes the nonrigorous introductory part of this section.

We start our rigorous analysis of the functional by proving that J is well-defined and continuously differentiable on the Banach space $X_s^{p'}$ where, from now on, $X_s^r:=L^r(\mathbb{R}^N,L_s^r(\mathbb{T}))$ for

$r \in (1,\infty)$ and $s \in \{1,..,5\}$. These spaces were introduced at the beginning of the paper. We will need the Hausdorff-Young inequality for Fourier series that we recall for the convenience of the reader.

Proposition 1 (Hausdorff-Young)**.** *Let $p \in [2,\infty]$. Then there is a $C>0$ such that*

$$\|f\|_{L^p(\mathbb{T})} \le C\|\hat{f}\|_{\ell^{p'}(\mathbb{Z})} \tag{10}$$

$$\|\hat{g}\|_{\ell^p(\mathbb{Z})} \le C\|g\|_{L^{p'}(\mathbb{T})} \tag{11}$$

whenever $\hat{f} \in \ell^{p'}(\mathbb{Z})$ and $g \in L^{p'}(\mathbb{T})$. Here, $\hat{g}(k) := \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}kt} g(t)\,\mathrm{d}t$ for $k \in \mathbb{Z}$.

The proofs of the following propositions are postponed to Section 4.

Proposition 2. *Assume (A1),(A2).*

(i) The operator $\mathscr{R} : X_s^{p'} \to X_s^p$ is well-defined, continuous, symmetric and compact.

(ii) The operator $Q^{-1/p}\mathscr{R} : X_s^{p'} \to L^q(\mathbb{R}^N, L_s^p(\mathbb{T}))$ is well-defined and continuous.

Using part (i) of the previous proposition we show that J satisfies the assumptions of the Symmetric Mountain Pass Theorem. This allows to conclude that there is an unbounded sequence of critical points. Those will provide the breather solutions U after inverting the formal passage to the dual variables from the beginning of this section.

Proposition 3. *Assume (A1),(A2),(A3). Then the functional $J : X_s^{p'} \to \mathbb{R}$ as in* (9) *is even, continuously differentiable and has the mountain pass geometry:*

(i) $J(0)=0$ and there are $r,\delta>0$ with $J(V) \ge \delta$ for all $V \in X_s^{p'}$ with $\|V\|_{p'} = r$.

(ii) There is an increasing sequence of linear subspaces $\mathfrak{W}^{(m)} \subseteq X_s^{p'}$ of dimension m, respectively, and radii $R_m > r$ such that $J(V)<0$ for all $V \in \mathfrak{W}^{(m)}$ with $\|V\|_{p'} > R_m$.

(iii) J satisfies the Palais-Smale condition.

The last of our preparatory results shows that each critical point of J indeed provides a 2π-periodic distributional beather solution as claimed in Theorem 1. Here we use part (ii) of Proposition 2.

Proposition 4. *Assume (A1),(A2) and let $V \in X_s^{p'}$ be a nontrivial critical point of J. Then the function $U := Q^{-1/p}\mathscr{R}[V] \in L^q(\mathbb{R}^N, L_s^p(\mathbb{T}))$ is well-defined and a nontrivial 2π-periodic distributional breather solution of the nonlinear wave equation* (2) *in the sense of* (4)*.*

We summarize the arguments mentioned above to prove our main result.

Proof of Theorem 1:
By Proposition 3 the functional J satisfies all assumptions of the Symmetric Mountain Pass Theorem [15, Corollary 7.23]. So there is an unbounded sequence of critical values of J. Since J maps bounded sets to bounded sets, we thus get an unbounded sequence of critical points $(V_j)_{j\in\mathbb{N}_0}$. By Proposition 4, the substitution $U_j := Q^{-1/p}|V_j|^{p'-2}V_j$ yields the asserted infinite sequence of distributional 2π-periodic breather solutions of the nonlinear wave equation (2) in $L^q(\mathbb{R}^N, L_s^p(\mathbb{T}))$. This sequence is unbounded because Hölder's inequality implies

$$\|Q\|_{\frac{q}{q-p}}^{1/p}\|U_j\|_q = \|Q^{1/p}\|_{\frac{pq}{q-p}}\|U_j\|_q \ge \|Q^{1/p}U_j\|_p = \|V_j\|_{p'}^{p'-1} \nearrow \infty \quad (j\to\infty).$$

□

4. Proofs of auxiliary results

Proof of Proposition 2.

Step 1: *Proof of (i) – Well-definedness and continuity.*
We show that the series in the definition of $\mathscr{R}[V]$ converges in $L^p(\mathbb{R}^N, L^p(\mathbb{T}))$ and that $\mathscr{R}$ preserves the time-symmetry, i.e., $\mathscr{R}(X_s^{p'}) \subset X_s^p$. To prove the first point we use the estimate

$$
\begin{aligned}
\|\mathcal{R}_k^Q w\|_{L^p(\mathbb{R}^N)} &= \|Q^{1/p}\mathcal{R}_k(Q^{1/p}w)\|_{L^p(\mathbb{R}^N)} \\
&\leq \|Q^{1/p}\|_{L^{\frac{pq}{q-p}}(\mathbb{R}^N)} \|\mathcal{R}_k(Q^{1/p}w)\|_{L^q(\mathbb{R}^N)} \\
&\overset{(A1)}{\leq} \|Q\|^{1/p}_{L^{\frac{q}{q-p}}(\mathbb{R}^N)} C(k^2+1)^{-\alpha/2}\|Q^{1/p}w\|_{L^{q'}(\mathbb{R}^N)} \\
&\leq \|Q\|^{1/p}_{L^{\frac{q}{q-p}}(\mathbb{R}^N)} \|Q^{1/p}\|_{L^{\frac{p'q'}{p'-q'}}(\mathbb{R}^N)} C(k^2+1)^{-\alpha/2}\|w\|_{L^{p'}(\mathbb{R}^N)} \\
&= \|Q\|^{2/p}_{L^{\frac{q}{q-p}}(\mathbb{R}^N)} C(k^2+1)^{-\alpha/2}\|w\|_{L^{p'}(\mathbb{R}^N)} \\
&\leq C_1(k^2+1)^{-\alpha/2}\|w\|_{L^{p'}(\mathbb{R}^N)} \qquad \text{where } C_1 := C\|Q\|^{2/p}_{L^{\frac{q}{q-p}}(\mathbb{R}^N)}.
\end{aligned} \tag{12}
$$

Next we determine a uniform upper bound for the corresponding finite serieses over finitely many modes $k \in \mathcal{I}_s$. For notational convenience this will be expressed as a sum over k:

$$
\begin{aligned}
\left\|\sum_k \mathrm{e}^{\mathrm{i}k\cdot}\ \mathcal{R}_k^Q\left[v_k\right]\right\|_p &= \left\|\left\|\sum_k \mathrm{e}^{\mathrm{i}k\cdot}\ \mathcal{R}_k^Q\left[v_k\right](x)\right\|_{L^p(\mathbb{T})}\right\|_{L^p(\mathbb{R}^N)} \\
&\overset{(10)}{\leq} \left\|\left(\sum_k |\mathcal{R}_k^Q\left[v_k\right]|^{p'}\right)^{1/p'}\right\|_{L^p(\mathbb{R}^N)} \\
&= \left\|\sum_k |\mathcal{R}_k^Q\left[v_k\right]|^{p'}\right\|^{1/p'}_{L^{p-1}(\mathbb{R}^N)} \\
&\overset{p>2}{\leq} \left(\sum_k \left\||\mathcal{R}_k^Q\left[v_k\right]|^{p'}\right\|_{L^{p-1}(\mathbb{R}^N)}\right)^{1/p'} \\
&= \left(\sum_k \left\|\mathcal{R}_k^Q\left[v_k\right]\right\|^{p'}_{L^p(\mathbb{R}^N)}\right)^{1/p'} \\
&\overset{(12)}{\leq} C_1 \left(\sum_k (k^2+1)^{-\frac{\alpha p'}{2}}\|v_k\|^{p'}_{L^{p'}(\mathbb{R}^N)}\right)^{1/p'}
\end{aligned}
$$

$$
\begin{aligned}
&= C_1 \left(\int_{\mathbb{R}^N} \sum_k (k^2+1)^{-\frac{\alpha p'}{2}} |v_k(x)|^{p'} \,\mathrm{d}x \right)^{1/p'} \\
&\overset{p>2}{\leq} C_1 \left(\int_{\mathbb{R}^N} \left(\sum_k (k^2+1)^{-\frac{\alpha p'}{2}\cdot\frac{p-1}{p-2}} \right)^{\frac{p-2}{p-1}} \left(\sum_k |v_k(x)|^p \right)^{\frac{p'}{p}} \,\mathrm{d}x \right)^{1/p'} \\
&\overset{(11)}{\leq} C_1 \left(\sum_k (k^2+1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p-2}{p}} \left\| \|V(\cdot,x)\|_{L^{p'}(\mathbb{T})} \right\|_{L^{p'}(\mathbb{R}^N)} \\
&= C_1 \left(\sum_k (k^2+1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p-2}{p}} \|V\|_{p'} .
\end{aligned}
$$

So, summing up, we have proved

$$
\left\| \sum_{|k|\leq k^*, k\in\mathcal{I}_s} \mathrm{e}^{\mathrm{i}k\,\cdot}\, \mathcal{R}_k^Q[v_k] \right\|_p \leq C_1 \left(\sum_{|k|\leq k^*, k\in\mathcal{I}_s} (k^2+1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p-2}{p}} \|V\|_{p'}, \tag{13}
$$

and this completes the proof since the sum converges if and only if $\frac{\alpha p}{p-2} > 1$, which holds by assumption. Therefore $\mathscr{R}[V]$ is a well-defined limit in $L^p(\mathbb{R}^N, L^p(\mathbb{T}))$ and

$$
\|\mathscr{R}[V]\|_p \leq C_1 \left(\sum_{k\in\mathcal{I}_s} (k^2+1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p-2}{p}} \|V\|_{p'} .
$$

Notice that $\mathscr{R}$ maps real-valued functions to real-valued functions because so do the operators $\mathcal{R}_k^Q$. So it remains to prove the time-symmetry preserving property of $\mathscr{R}$, i.e., $\mathscr{R}(X_s^{p'}) \subset X_s^p$. Recall that elements $V \in X_s^{p'}$ are real-valued by assumption and thus satisfy $v_k = \overline{v_{-k}}$ for all $k \in \mathbb{Z}$ and all $s = 1, \ldots, 5$.

- For $s = 1$ there is nothing to prove.
- For $s = 2$ any $V \in X_s^{p'}$ is even in time. Equivalently, all Fourier modes $v_k = v_{-k}$ are real-valued. This is true also for $\mathscr{R}V$ because $\mathcal{R}_k = \mathcal{R}_{-k}$ implies

$$
[\mathscr{R}V]_k = \mathcal{R}_k^Q[v_k] = \mathcal{R}_{-k}^Q[v_{-k}] = [\mathscr{R}V]_{-k}, \quad [\mathscr{R}V]_k = \mathcal{R}_k^Q[v_k] = \mathcal{R}_k^Q[\overline{v_k}] = \overline{\mathcal{R}_k^Q[v_k]} = \overline{[\mathscr{R}V]_k}.
$$

- For $s = 3$ any $V \in X_s^{p'}$ is odd in time, i.e., the zero mode $v_0 = 0$ vanishes and the other Fourier modes $v_k = -v_{-k}$ are purely imaginary. Again, this is true also for $\mathscr{R}V$ because the zero mode does not occur in $\mathcal{I}_s$ and

$$
[\mathscr{R}V]_k = \mathcal{R}_k^Q[v_k] = \mathcal{R}_{-k}^Q[-v_{-k}] = -[\mathscr{R}V]_{-k}, \quad [\mathscr{R}V]_k = \mathcal{R}_k^Q[v_k] = \mathcal{R}_k^Q[-\overline{v_k}] = -\overline{[\mathscr{R}V]_k}.
$$

- For $s = 4$ any $V \in X_s^{p'}$ is π-periodic, i.e., the modes with odd k vanish. Since $\mathcal{I}_4 = 2\mathbb{Z}$ this is true as well for $\mathscr{R}V$.
- For $s = 5$ any $V \in X_s^{p'}$ is π-antiperiodic, i.e., the modes with even k vanish. Since $\mathcal{I}_5 = 2\mathbb{Z} + 1$ this is true as well for $\mathscr{R}V$.

<u>STEP 2</u>: *Proof of (i) – Symmetry and compactness.*
The symmetry of $\mathscr{R}$, which means

$$\int_{\mathbb{T}\times\mathbb{R}^N} \mathscr{R}[V](t,x)W(t,x)\ \mathrm{d}(t,x) = \int_{\mathbb{T}\times\mathbb{R}^N} V(t,x)\mathscr{R}[W](t,x)\ \mathrm{d}(t,x)$$

for all $V, W \in X_s^{p'}$, follows from the the continuity of $\mathscr{R} : X_s^{p'} \to X_s^p$ and the symmetry of $\mathcal{R}_k^Q$. We now turn to the proof of compactness. Assume that $(V^{(n)})_n$ is a bounded sequence in $X_s^{p'}$ with $\|V^{(n)}\|_{p'} \le C_V$ for all $n \in \mathbb{N}$. We aim to show that a subsequence of $(\mathscr{R}[V^{(n)}])_n$ converges in $X_s^{p'}$. Here, for almost all $t \in \mathbb{T}$ and $n \in \mathbb{N}$,

$$\mathscr{R}[V^{(n)}](t,\,\cdot\,) = \sum_{k\in\mathcal{I}_s} \mathrm{e}^{\mathrm{i}kt}\,\mathcal{R}_k^Q[v_k^{(n)}] \qquad \text{in } L^p(\mathbb{R}^N).$$

From Hölder's inequality we infer

$$\left\|v_k^{(n)}\right\|_{L^{p'}(\mathbb{R}^N)} = \left\|\frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}kt}\,V^{(n)}(t,\,\cdot\,)\,\mathrm{d}t\right\|_{L^{p'}(\mathbb{R}^N)} \le (2\pi)^{-\frac{1}{p'}}\ \left\|V^{(n)}\right\|_{p'} \le C_V.$$

So all sequences $(v_k^{(n)})_n$ are bounded in $L^{p'}(\mathbb{R}^N)$. The compactness of $\mathcal{R}_k^Q$ from (A2) combined with a standard diagonalization technique provides $y_k \in L^p(\mathbb{R}^N), k \in \mathbb{Z}$, and a subsequence $(v_k^{(n_j)})_j$ with

$$\forall\, k \in \mathcal{I}_s \qquad \mathcal{R}_k^Q[v_k^{(n_j)}] \to y_k \qquad \text{in } L^p(\mathbb{R}^N) \text{ as } j \to \infty. \tag{14}$$

We claim that

$$(\mathscr{R}[V^{(n_j)}])_j \to \sum_{k\in\mathcal{I}_s} \mathrm{e}^{\mathrm{i}k\,\cdot}\ y_k \quad \text{in } X_s^p. \tag{15}$$

That the latter function indeed belongs to X_s^p follows from

$$\begin{aligned}\left\|\sum_k \mathrm{e}^{\mathrm{i}k\,\cdot}\ y_k\right\|_p &= \lim_{j\to\infty}\left\|\sum_k \mathrm{e}^{\mathrm{i}k\,\cdot}\ \mathcal{R}_k^Q[v_k^{(n_j)}]\right\|_p \\ &\overset{(13)}{\le} C_1\left(\sum_k (k^2+1)^{-\frac{\alpha p}{2(p-2)}}\right)^{\frac{p-2}{p}} \limsup_{j\to\infty}\left\|V^{(n_j)}\right\|_{p'} \\ &\le C_1 C_V\left(\sum_{k\in\mathcal{I}_s} (k^2+1)^{-\frac{\alpha p}{2(p-2)}}\right)^{\frac{p-2}{p}} < \infty.\end{aligned}$$

Combining this with (13) and (14), we obtain (15) using a 3ε-argument.

<u>STEP 3</u>: *Proof of (ii).*
We essentially repeat the estimate from the first step where the exponent q replaces p in the spatial Lebesgue norm in order to take the missing factor $Q^{1/p}$ into account. Given that

$q \geq p'$ and that our summation is over finitely many indices only, we obtain

$$\begin{aligned}
\left\| \left\| \sum_k \mathrm{e}^{\mathrm{i}kt} \mathcal{R}_k[Q^{1/p} v_k](x) \right\|_{L^p(\mathbb{T})} \right\|_{L^q(\mathbb{R}^N)} &\overset{q \geq p'}{\leq} \left(\sum_k \left\| \mathcal{R}_k[Q^{1/p} v_k] \right\|_{L^q(\mathbb{R}^N)}^{p'} \right)^{1/p'} \\
&\leq C \left(\sum_k (k^2+1)^{-\frac{\alpha p'}{2}} \left\| Q^{1/p} v_k \right\|_{L^{q'}(\mathbb{R}^N)}^{p'} \right)^{1/p'} \\
&\leq C \left(\sum_k (k^2+1)^{-\frac{\alpha p'}{2}} \| Q^{1/p} \|_{\frac{p'q'}{p'-q'}}^{p'} \, \|v_k\|_{L^{p'}(\mathbb{R}^N)}^{p'} \right)^{1/p'} \\
&\leq C \|Q\|_{\frac{q}{q-p}}^{1/p} \left(\sum_k (k^2+1)^{-\frac{\alpha p'}{2}} \|v_k\|_{L^{p'}(\mathbb{R}^N)}^{p'} \right)^{1/p'} \\
&\overset{\text{step 1}}{\leq} C \|Q\|_{\frac{q}{q-p}}^{1/p} \left(\sum_k (k^2+1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p-2}{p}} \|V\|_{p'}.
\end{aligned}$$

Since the sum converges, we get the result. □

Proof of Proposition 3.
We prove that the functional

$$J : L_s^{p'}(\mathbb{T} \times \mathbb{R}^N) \to \mathbb{R}, \quad J(V) := \frac{1}{p'} \int_{\mathbb{T}\times\mathbb{R}^N} |V|^{p'} \, \mathrm{d}(t,x) - \frac{1}{2} \int_{\mathbb{T}\times\mathbb{R}^N} V \mathscr{R}[V] \, \mathrm{d}(t,x)$$

satisfies the assumptions of the Symmetric Mountain Pass Theorem. It is straightforward to deduce from Proposition 2 that J is well-defined, even and of class C^1.

(i) Assuming $\|V\|_{p'} = r$, we estimate using $C_R := \|\mathscr{R}\|_{p' \to p} < \infty$ and get

$$\begin{aligned}
J(V) &= \frac{1}{p'} \int_{\mathbb{T}\times\mathbb{R}^N} |V|^{p'} \, \mathrm{d}(t,x) - \frac{1}{2} \int_{\mathbb{T}\times\mathbb{R}^N} V \mathscr{R}[V] \, \mathrm{d}(t,x) \\
&\geq \frac{1}{p'} \|V\|_{p'}^{p'} - \frac{C_R}{2} \|V\|_{p'}^2 \\
&= r^{p'} \left(\frac{1}{p'} - \frac{C_R}{2} r^{2-p'} \right).
\end{aligned}$$

Hence, the claim (i) holds for $r = (C_R p')^{-1/(2-p')}$ and $\delta = r^{p'}/2p' > 0$.

(ii) According to (A3) we find $\omega_k \in L^{p'}(\mathbb{R}^N)$ such that w.l.o.g.

$$\int_{\mathbb{R}^N} \omega_k \mathcal{R}_k^Q[\omega_k] \, \mathrm{d}x = \frac{2}{\pi} \quad \text{for all } k \in \mathcal{I}_s.$$

With that, we choose for $k \in \mathcal{I}_s$

$$V_k(t,x) := w_k(x) T_k(t) := \begin{cases} w_k(x) \cos(kt) & \text{if } s \in \{1,2,4\}, \\ w_k(x) \sin(kt) & \text{if } s \in \{3,5\}. \end{cases}$$

This choice guarantees $V_k \in X_s^{p'}$ for all $k \in \mathcal{I}_s$. Moreover,

$$\int_{\mathbb{T}\times\mathbb{R}^N} V_{k'}\, \mathscr{R}[V_k]\,\mathrm{d}(t,x) = \int_{\mathbb{T}} T_{k'}(t)T_k(t)\,\mathrm{d}t \cdot \int_{\mathbb{R}^N} \omega_{k'}\, \mathcal{R}_k^Q[\omega_k]\,\mathrm{d}x \begin{cases} = 0 & \text{if } k \neq k', \\ = 2 & \text{if } k = k'. \end{cases}$$

So the V_k are linearly independent. Choosing nested subsets $\mathcal{I}_s^j \subset \mathcal{I}_s$ with j elements and $\mathfrak{W}_j := \operatorname{span}\{V_k : k \in \mathcal{I}_s^j\}$ we get $\dim \mathfrak{W}_j = j$. For any fixed $j \in \mathbb{N}$, equivalence of norms provides a constant $c_j > 1$ with

$$\frac{1}{c_j}\left(\sum_{k\in\mathcal{I}_s^j} \beta_k^2\right)^{1/2} \leq \left\|\sum_{k\in\mathcal{I}_s^j} \beta_k V_k\right\|_{p'} \leq c_j \left(\sum_{k\in\mathcal{I}_s^j} \beta_k^2\right)^{1/2} \quad \text{whenever } \beta_k \in \mathbb{R}, k \in \mathcal{I}_s^j.$$

For $R > r$ and some arbitrary element $V = \sum_{k\in\mathcal{I}_s^j} \beta_k V_k \in \mathfrak{W}_j$ with $\|V\|_{p'} = R$, we obtain the estimate

$$\begin{aligned} J(V) &= \frac{1}{p'}\int_{\mathbb{T}\times\mathbb{R}^N} |V|^{p'}\,\mathrm{d}(t,x) - \frac{1}{2}\int_{\mathbb{T}\times\mathbb{R}^N} V\,\mathscr{R}[V]\,\mathrm{d}(t,x) \\ &= \frac{1}{p'}\int_{\mathbb{T}\times\mathbb{R}^N} |V|^{p'}\,\mathrm{d}(t,x) - \frac{1}{2}\sum_{k,k'\in\mathcal{I}_s^j} \beta_k\beta_{k'}\int_{\mathbb{T}\times\mathbb{R}^N} V_{k'}\,\mathscr{R}[V_k]\,\mathrm{d}(t,x) \\ &= \frac{1}{p'}\cdot R^{p'} - \sum_{k\in\mathcal{I}_s^j}\beta_k^2 \\ &\leq \frac{1}{p'}\cdot R^{p'} - \frac{1}{c_j^2}\cdot R^2. \end{aligned}$$

Since $p' < 2$, we thus conclude for $R_j := \max\left\{r, \left(c_j^2/p'\right)^{1/(2-p')}\right\}$ that $J(V) < 0$ whenever $V \in \mathfrak{W}_j$ with $\|V\|_{p'} > R_j$.

(iii) Take any Palais-Smale sequence $(V_n)_n$ for J, that is, $V_n \in X_s^{p'}$ with

$$J'(V_n) \to 0 \quad \text{in } \left(X_s^{p'}\right)' = X_s^p, \qquad J(V_n) \to c$$

where $c > 0$ denotes the mountain pass level. We claim that the sequence $(V_n)_n$ is bounded. Indeed, assuming otherwise, the identity

$$J'(V_n)[V_n] - 2J(V_n) = \left(1 - \frac{2}{p'}\right)\int_{\mathbb{T}\times\mathbb{R}^N} |V_n|^{p'}\,\mathrm{d}(t,x)$$

leads in the limit $n \to \infty$ to the contradictory statement

$$0 = \limsup_{n\to\infty} \frac{J'(V_n)[V_n] - 2J(V_n)}{\|V_n\|_{p'}} = \limsup_{n\to\infty}\left(1 - \frac{2}{p'}\right)\|V_n\|_{p'}^{p'-1} = -\infty.$$

Hence, we may assume w.l.o.g. that $V_n \rightharpoonup V$ weakly in $X_s^{p'}$ for some $V \in X_s^{p'}$. Due to the compactness of $\mathscr{R}$ (see Proposition 2), this implies $\mathscr{R}[V_n] \to \mathscr{R}[V]$ strongly in $L_s^p(\mathbb{T}\times\mathbb{R}^N)$.

Hence, we obtain

$$\int_{\mathbb{T}\times\mathbb{R}^N} V_n \mathscr{R}[V_n]\,\mathrm{d}(t,x) \to \int_{\mathbb{T}\times\mathbb{R}^N} V\mathscr{R}[V]\,\mathrm{d}(t,x).$$

As in the proof of [13, Lemma 5.2] we obtain $V_n \to V$ in the strong sense by uniform convexity of $L^{p'}(\mathbb{T}\times\Omega)$ and the statement is proved. □

Proof of Proposition 4.
We consider a nontrivial critical point $V \in L^{p'}_s(\mathbb{T}\times\mathbb{R}^N)$ of the functional J. Since $\mathscr{R}$ is symmetric by Proposition 2 (i), the Euler-Lagrange equation reads

$$|V|^{p'-2}V = \mathscr{R}[V] \qquad \text{in } X^p_s = L^p(\mathbb{R}^N, L^p_s(\mathbb{T})). \tag{16}$$

From Proposition 2 (ii) and (16) we infer $U := (Q^{-1/p}\mathscr{R})[V] \in L^q(\mathbb{R}^N, L^p_s(\mathbb{T}))$. We will use $Q^{1/p}U = Q^{1/p}\cdot(Q^{-1/p}\mathscr{R})[V] = \mathscr{R}[V] = |V|^{p'-2}V$ and thus

$$Q|U|^{p-2}U = Q^{1/p}\cdot|Q^{1/p}U|^{p-2}Q^{1/p}U = Q^{1/p}V.$$

So it remains to verify

$$\int_{\mathbb{T}\times\mathbb{R}^N} Q|U|^{p-2}U\,\Phi\,\mathrm{d}(t,x) = \int_{\mathbb{T}\times\mathbb{R}^N} U\,(\partial_t^2-\mathcal{L})\Phi\,\mathrm{d}(t,x)$$

for all $C_c^\infty(\mathbb{R}^N, C^\infty(\mathbb{T}))$. We first verify the above identity for (real-valued) test functions of the form $\Phi(t,x) := \sum_k \mathrm{e}^{-\mathrm{i}kt}\phi_k(x) \in X^{p'}_s$ where again the sum is taken over finitely many $k\in\mathcal{I}_s$ and $\phi_l, \phi_{-l} = \overline{\phi_l} \in C_c^\infty(\mathbb{R}^N)$. We then have

$$\begin{aligned}
&\int_{\mathbb{T}\times\mathbb{R}^N} Q(x)|U(t,x)|^{p-2}U(t,x)\Phi(t,x)\,\mathrm{d}(t,x)\\
&\quad= \int_{\mathbb{T}\times\mathbb{R}^N} Q(x)^{1/p}V(t,x)\Phi(t,x)\,\mathrm{d}(t,x)\\
&\quad= \sum_k \int_{\mathbb{R}^N} Q(x)^{1/p}\phi_k(x)\left[\int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}kt}\,V(t,x)\,\mathrm{d}t\right]\mathrm{d}x\\
&\quad= 2\pi\sum_k \int_{\mathbb{R}^N} \phi_k(x)Q(x)^{1/p}v_k(x)\,\mathrm{d}x\\
&\quad\overset{(A2)}{=} 2\pi\sum_k \int_{\mathbb{R}^N} (-\mathcal{L}-k^2)\phi_k(x)\;\mathcal{R}_k[Q^{1/p}v_k](x)\,\mathrm{d}x\\
&\quad= 2\pi\int_{\mathbb{R}^N}\sum_k Q(x)^{-1/p}\mathcal{R}^Q_k[v_k](x)\;(-\mathcal{L}-k^2)\phi_k(x)\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^N}\sum_{k\in\mathcal{I}_s} Q(x)^{-1/p}\mathcal{R}^Q_k[v_k](x)\;(-\mathcal{L}-k^2)\left[\int_{\mathbb{T}}\mathrm{e}^{\mathrm{i}kt}\Phi(t,x)\,\mathrm{d}t\right]\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^N}\sum_{k\in\mathcal{I}_s} Q(x)^{-1/p}\mathcal{R}^Q_k[v_k](x)\;\left[\int_{\mathbb{T}} -\mathrm{e}^{\mathrm{i}kt}\mathcal{L}\left[\Phi(t,\cdot)\right](x) + \partial_t^2\left[\mathrm{e}^{\mathrm{i}kt}\right]\Phi(t,x)\,\mathrm{d}t\right]\mathrm{d}x.
\end{aligned}$$

We now integrate by parts. Since $\Phi(\,\cdot\,,x)$ is periodic, the boundary terms in $\int_0^{2\pi}\Phi\partial_t^2\left[\mathrm{e}^{\mathrm{i}kt}\right]\mathrm{d}t = \left[\mathrm{i}k\Phi\cdot\mathrm{e}^{\mathrm{i}kt}+(\partial_t\Phi)\cdot\mathrm{e}^{\mathrm{i}kt}\right]_0^{2\pi}+\int_0^{2\pi}\mathrm{e}^{\mathrm{i}kt}\partial_t^2\Phi\,\mathrm{d}t$ vanish for a.e. $x\in\mathbb{R}^N$. So we get

$$\begin{aligned}
&\int_{\mathbb{T}\times\mathbb{R}^N} Q(t,x)|U(t,x)|^{p-2}U(t,x)\,\Phi(t,x)\,\mathrm{d}(t,x)\\
&\quad=\int_{\mathbb{R}^N}\sum_{k\in\mathcal{I}_s} Q(x)^{-1/p}\mathcal{R}_k^Q[v_k](x)\,\left[\int_{\mathbb{T}}\mathrm{e}^{\mathrm{i}kt}\left(\partial_t^2-\mathcal{L}\right)\Phi\,\mathrm{d}t\right]\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^N}\sum_{k\in\mathcal{I}_s}\int_{\mathbb{T}}\mathrm{e}^{\mathrm{i}kt}Q(x)^{-1/p}\mathcal{R}_k^Q[v_k](x)\,\left(\partial_t^2-\mathcal{L}\right)\Phi\,\mathrm{d}t\,\mathrm{d}x\\
&\quad=\int_{\mathbb{T}\times\mathbb{R}^N} Q(x)^{-1/p}\sum_{k\in\mathcal{I}_s}\mathrm{e}^{\mathrm{i}kt}\mathcal{R}_k^Q[v_k](x)\,\left(\partial_t^2-\mathcal{L}\right)\Phi\,\mathrm{d}(t,x)\\
&\quad=\int_{\mathbb{T}\times\mathbb{R}^N} Q^{-1/p}\mathscr{R}[V]\,(\partial_t^2-\mathcal{L})\Phi\,\mathrm{d}(t,x)\\
&\quad\overset{(16)}{=}\int_{\mathbb{T}\times\mathbb{R}^N} Q^{-1/p}|V|^{p'-2}V\,(\partial_t^2-\mathcal{L})\Phi\,\mathrm{d}(t,x)\\
&\quad=\int_{\mathbb{T}\times\mathbb{R}^N} U\,(\partial_t^2-\mathcal{L})\Phi\,\mathrm{d}(t,x).
\end{aligned}$$

Now we extend this identity to arbitrary $\Phi\in C_c^\infty(\mathbb{R}^N,C^\infty(\mathbb{T}))$. From above, we obtain that the identity (4) holds for Φ replaced by real-valued test functions of the form

$$\sum_{k\in\mathcal{I}_s,|k|\le k^*}\mathrm{e}^{-\mathrm{i}kt}\,\phi_k(x)\quad\text{where}\quad\phi_k(x)=\frac{1}{2\pi}\int_{\mathbb{T}}\mathrm{e}^{\mathrm{i}kt}\,\Phi(t,x)\,\mathrm{d}t.$$

Since the functions $U(\cdot,x)$ and $Q|U(\cdot,x)|^{p-2}U(\cdot,x)$ have the symmetry in time indexed by s for almost all $x\in\mathbb{R}^N$, they are $L^2(\mathbb{T})$-orthogonal to the modes $\mathrm{e}^{-\mathrm{i}kt}$ with $k\in\mathbb{Z}\setminus\mathcal{I}_s$. So we see that (4) actually holds for Φ replaced by

$$\sum_{k\in\mathbb{Z},|k|\le k^*}\mathrm{e}^{-\mathrm{i}kt}\,\phi_k(x).$$

It remains to pass to the limit $k^*\to\infty$ because of $\Phi(x,t)=\sum_{k\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}kt}\,\phi_k(x)$. Here we use that $\Phi(\cdot,t)$ and all the ϕ_k have support contained in some fixed compact set $K\subseteq\mathbb{R}^N$. Moreover, since $\Phi,\mathcal{L}\Phi$ are smooth in time, we have

$$\|\phi_k\|_\infty+\|\mathcal{L}\phi_k\|_\infty\le(k^2+1)^{-100}\qquad\text{for all }k\in\mathbb{Z}.$$

So the Fourier series for $(\partial_t^2-\mathcal{L})\Phi$ converges uniformly on $\mathbb{R}^N$, which implies

$$\int_{\mathbb{T}\times\mathbb{R}^N} Q(x)|U|^{p-2}U\Phi\,\mathrm{d}(t,x)=\int_{\mathbb{T}\times\mathbb{R}^N} U\,(\partial_t^2-\mathcal{L})\Phi\,\mathrm{d}(t,x).$$

This finishes the proof.

$\square$

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